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Selection of the Guidance Variable for a Re-entry Vehicle

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The use of a dynamic model employing an independent variable of evolution other than time lessens the impact of certain deficiencies possessed by linear-quadratic regulators when used in trajectory-following problems. For those situations in which there are several candidates with the requisite properties of an independent variable, this paper provides an index of dynamic performance which can be used to order the alternatives. It is shown that for an important class of trajectories, variables exist that provide good closed-loop behavioral properties along with a simple guidance-law implementation. The advantage which accrues to the use of such a variable in place of time is illustrated by an example.

Introduction

LINEAR-quadratic (LQ) regulator theory provides a method for synthesizing guidance laws which will cause a re-entry vehicle to traverse a specified path and impact the ground near a designated point. The command is generated as the sum of two components. The first, an open-loop actuating signal, yields the desired vehicle trajectory in the absence of exogenous disturbances. The second component is a linear-feedback regulator which reduces the system sensitivity to untoward influences on the vehicle. It is well known that, subject to certain technical qualifications, the closed-loop system is stable about the nominal trajectory and has additional desirable attributes.

An aerodynamically controlled re-entry vehicle has those properties which make the application of LQ theory feasible, but as pointed out in Ref. 1, it has them "weakly." Specifically, for the short flight time between re-entry and impact the advantageous asymptotic characteristics of the LQ regulator may be lost. For example, the asymptotic stability of the closed-loop system does not guarantee any perceptible diminution of the magnitude of trajectory errors during the interval over which guidance is accomplished. Indeed, unless the weighting matrices of the performance index are selected judiciously, initial trajectory errors will have significant influence on impact errors.

An inherent failing of an LQ regulator is its tendency to "overcontrol" the vehicle. The guidance law is intended to cause the vehicle to follow a specified path in space, but the LQ regulator seeks to do more than this. Because time is the independent variable in the synthesis algorithm, not only does the regulator try to follow the desired path, but it tries to follow this path at a particular rate. This latter task is not only unnecessary, but for reasons discussed in Ref. 1, it may prove to be quite difficult to perform. By posing the LQ regulator problem in the classical way, the regulator is implicitly given a perverse assignment which taxes its available resources.

The intrinsic problems associated with using time-based guidance laws in this and similar applications were perceived early on. In 1968 Speyer and Bryson,[2] studied a guidance problem with state-equality constraints at termination. Such constraints give rise to infinite gains at the terminal time. It was observed that this will lead to implementational difficulties if the feedback gains use clock time as a lookup variable. If, for example, the vehicle is slowed by some exogenous influence, the high gains will be used prematurely. Speyer and Bryson, therefore, proposed an alternative. The gain tables were computed using a time-based model of the vehicle, but they were entered into the tables "at an 'index time' determined so that time-to-go on the neighboring path is the same as the time-to-go on the nominal path." An estimate of time-to-go is provided in Ref. 2 along with the resulting state-feedback law.

Powers[3] investigated a problem with similar motivation. He was concerned with finding the most effective utilization of neighboring optimum guidance. Here again a time-base model was used to derive a guidance law. To avoid sensitivity to time translations, Powers proposed that guidance be implemented on the basis of computing trajectory errors as deviations from the trajectory point "closest" to the observed stated. He noted that the applicability of the technique of neighboring guidance is strongly dependent upon the definition of the "closest" in this context.

The work presented in Ref. 4 took a somewhat different approach to the readily acknowledged problem of performance sensitivity when guidance is accomplished with a "clock-time" implementation. Instead of deriving the regulator equations using a time-based model and then eliminating this time dependence during the realization phase of synthesis, a trajectory variable was used in place of time as the "independent" guidance variable in the vehicle model and the resulting regulator had no explicit "clock-time" dependence. In Ref. 4 a regulator of this generic type was delineated which retained the implementational simplicity of the LQ regulator while ameliorating some of its undesirable transient characteristics.

The work in this latter reference assumed that the independent guidance variable was given initially when, in fact, there may be several candidate variables with the requisite properties. In many applications the choice of the best variable of dynamic evolution becomes one of the design tasks. In this paper this problem is studied in the context of a specific class of missions for an aerodynamically controlled re-entry vehicle. A study of the relative advantages of an easily parameterized set of guidance laws is presented. A

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comprehensive trajectory simulation provides insight into the significance of a docility index used to order the associated class of independent variables.

Guidance-Law Description

The dynamic behavior of the aerodynamically controlled re-entry vehicle is described by a nonlinear differential equation of the form

$$\dot{\xi}(t)=f(\xi(t),\omega(t)) \qquad (t_0\le t\le t_f)$$

$$\xi(t_0)=\xi_0 \tag{1}$$

where $\xi(t)$ is the n-dimensional state vector and $\omega(t)$ is the actuating signal. Let x_n be the solution to Eq. (1) corresponding to the nominal open-loop actuating signal u_n and initial condition $x_n(t_0)$. It will be assumed that (x_n, u_n) represents the desired response of the vehicle and any deviation from (x_n, u_n) is to be made as small as possible.

The nominal trajectory provides only an idealization of the actual vehicle motion. For a variety of reasons the actual trajectory variables (x_p, u_p) may differ from their nominal values. The perturbation variables (x, u) are defined by

$$x(t)=x_p(t)-x_n(t) \qquad u(t)=u_p(t)-u_n(t) \tag{2}$$

The basic feedback regulator synthesis problem is that of selecting a control policy which generates u in such a way that (x, u) is made small.

Given that: the primary source of exogenous disturbance is the initial error in Eq. (1), the quality of the guidance law is measured by the quadratic performance index J,

$$J=x(t_f)'Px(t_f)+\int_{t_0}^{t_f}(x'Qx+u'Ru)\mathrm{d}\tau$$

$$(P,Q\ge 0,\ R>0) \tag{3}$$

If f is sufficiently smooth and (x, u) sufficiently small, and full-state feedback is permitted, it is well known that the solution to the regulator synthesis problem would be the LQ regulator:

$$u=-R^{-1}G'Kx$$

where

$$\dot{K}=-F'K-KF+KGR^{-1}GK-Q$$

$$K(t_f)=P$$

$$F=\frac{\partial f}{\partial x}\bigg|_{(x_n,\,u_n)} \qquad G=\frac{\partial f}{\partial u}\bigg|_{(x_n,\,u_n)} \tag{4}$$

If the perturbation equations satisfy certain technical restrictions, the LQ regulator has desirable stability and sensitivity properties that justify its use even when J is not thought to be a valid performance measure.

Surprisingly, the guidance law described by Eq. (4) yields very poor performance for a large class of (P, Q, R) weights. The reason for this is discussed in detail in Ref. 1, but suffice it to say that the "weak" controllability of the $[F, G]$ matrix associated with the re-entry vehicle leads to anomalous closed-loop dynamic response. The error variables defined by Eq. (2) aid in understanding the underlying case of this deleterious behavior. Suppose at time t_1

$$x_p(t_1)=x_n(t_1^*) \qquad (t_1\ne t_1^*) \tag{5}$$

i.e., the actual trajectory at t_1 is on the desired trajectory, but it has reached the desired point at an incorrect time. This is an inconsequential error in the application envisioned here, but the regulator derived on the basis of Eq. (4) would perceive the time-translation error of Eq. (5) as a state error and would attempt to eliminate it. For reasons presented in Ref. 1 this leads to a significant deterioration in performance.

Some of the disadvantageous behavorial properties of the guidance law given in Eq. (4) can be avoided if a trajectory variable is used in place of time as the variable of system evolution. Suppose $0<M_1<\dot{x}_{p1}<M_2$‡; $x_{p1}(t_0)\ge 0$. Then x_{p1} is a pseudotime variable and can replace time as the independent variable in Eq. (1);

$$\frac{\mathrm{d}x_p}{\mathrm{d}x_{p1}}=\frac{f(x_p,u_p)}{f_1(x_p,u_p)}=f_r(x_p,u_p) \tag{6}$$

To call x_{p1} "independent" is a misnomer in so far as x_{p1} is subject to the influences of the actuating signal and the exogenous variables. Nevertheless, this usage will be continued to contrast the role of x_{p1} with other components of x_p.

Viewing Eq. (6) as the dynamic equation of the vehicle, errors are not computed by comparing x_p and x_n at a fixed point in time, but rather by comparing x_p and x_n at a fixed value of x_{p1}. Denote the perturbation variables in this coordinate system by (x_r, u_r), where

$$x_r=x_p(x_{p1})-x_n(x_{p1})$$

$$u_r=u_p(x_{p1})-u_p(x_{p1}) \tag{7}$$

It is shown in Ref. 4 that (x_r, u_r) satisfies the differential equation

$$\frac{\mathrm{d}x_r}{\mathrm{d}x_{p1}}=F_rx_r+G_ru_r \qquad x_{p1}(t_0)\le x_{p1}\le x_{p1}(t_f)$$

$$F_r=\frac{\partial f_r}{\partial x}\bigg|_{(x_n,\,u_n)} \qquad G_r=\frac{\partial f_r}{\partial u}\bigg|_{(x_n,\,u_n)} \tag{8}$$

In the (x_r, u_r) coordinate system the LQ regulator can be delineated

$$u=-R_r^{-1}G_r'K_rx_r \tag{9}$$

where the matrices in Eq. (9) are defined in ways identical to those in Eq. (4), with the subscript r serving to distinguish relevant quantities in the (x_r, u_r) coordinate system.

It is tacitly assumed in Eq. (6) that the independent variable is given initially. In fact, this variable may be chosen by the analyst subject to the exigencies imposed by its inherent structural constraints and implementational considerations. To see how this design flexibility can be used to advantage, consider the class of independent variables generated by linear transformations of the state space.

Let $\{T(\zeta)\}$ be a set of nonsingular $n\times n$ matrices parameterized by the scalar index ζ. Denote by $x_n(\zeta)$ the nominal trajectory under the transformation $T(\zeta)$;

$$x_n(\zeta)=T(\zeta)x_n \tag{10}$$

Restricting attention to those values of ζ for which $x_{p1}(\zeta)=[T(\zeta)x_p]_1$ is monotone, a class of systems given generically by Eq. (6) can be displayed; i.e.,

$$\frac{\mathrm{d}x_p(\zeta)}{\mathrm{d}x_{p1}(\zeta)}=f_r(x_p,u_p;\zeta) \tag{11}$$

Corresponding to a specific value of ζ there is a dynamic equation of the re-entry vehicle, Eq. (11). To synthesize the best regulator, the analyst must select a permissible value of ζ in such a way as to make the closed-loop system perform in the best possible way. Observe that different system representations are being compared and a germane "docility" index which would expedite this comparison would be useful.

‡If x_s is a vector, x_{sj} is its jth component. For vectors x and f, x_1 and f_1 are their respective first components.

Before defining a docility index for this system, a few observations are apropos. Consider the nonlinear system of Eq. (1) with its perturbation equation characterized by the $[F, G]$ matrix of Eq. (4). If the perturbation equation is controllable at time t_1, it is well known that any observed error $x(t_1)$ can be eliminated at time $t_2 > t_1$ with minimum expenditure of control energy E given by Ref. 5§

$$E[x(t_1); t_1, t_2] = (t_2 - t_1)^{-1} \int_{t_1}^{t_2} u^2(\tau)\mathrm{d}\tau$$
$$= (t_2 - t_1)^{-1} x'(t_1) W^{-1}(t_1, t_2) x(t_1) \qquad (12)$$

where

$$W(t_1, t_2) = \int_{t_1}^{t_2} \Phi(t_1, \tau) G(\tau) G'(\tau) \Phi'(t_1, \tau)\mathrm{d}\tau \qquad (13)$$

and Φ is the transition matrix associated with F. Let $\{\lambda_i(t_1, t_2)\}$ be the positive eigenvalues of W arrayed in descending order and let $\{\eta_i(t_1, t_2)\}$ be the associated set of eigenvectors ordered in conformity with $\{\lambda_i(t_1, t_2)\}$. Let

$$\eta_i = \lim_{t_2 \to t_1} \eta_i(t_1, t_2) \qquad (14)$$

It is shown in the Appendix that if $|t_2 - t_1|$ is small and u is scalar valued,

$$E(\eta_i; t_1, t_2) = \alpha_i |t_2 - t_1|^{-2i} + \text{higher order terms} \quad (i = 1, \ldots n) \qquad (15)$$

The equations for the individual α_i are given in the Appendix [see Eq. (A14)].

Equation (15) indicates that there is a natural decomposition of the state space into a set of orthogonal directions characterized by the difficulty with which initial errors along each of the directions can be eliminated. For example, an error in the "easy" direction, η_1, can be eradicated by an actuating signal expending energy proportional to $|t_2 - t_1|^{-2}$. Observe that as the time interval over which control is accomplished decreases, the amplitude of the actuating signal is perforce increased. The asymptotic behavior of E represents the confluence of these antithetical limits.

In Eq. (15) the time factor $|t_2 - t_1|^k$ is a fixed scaling, but the set $\{\alpha_i\}$ gives a local indication of the system's ability to steer out errors. Because only local energy figures have been derived, the relative merits of different systems would be expected to vary with time. The way in which these sets are used to form a performance index depends upon the specific objective envisioned for the vehicle. An example of such an index is given in the following section.

Selection of the Independent Variable

Different independent variables, or equivalently different choices of ζ, give rise to different behavioral characteristics for the vehicle in the coordinate system associated with the regulator. To put the problem of selecting the best value of ζ on a rational basis, some criterion of choice is required. In the Appendix, one such criterion is discussed. In lieu of trying to minimize the individual values of the α_i in Eq. (15) a docility index H is defined as follows:¶

$$H(\zeta) = \prod_{i=1}^{n-1} \alpha_i(\zeta) \qquad (16)$$

This function orders the set of permissible independent variables. The value of ζ is sought which minimizes H;

$$H(\zeta^*) = \inf_{\zeta} H(\zeta) \qquad (17)$$

Note the $\{\alpha_i\}$ are coefficients which characterize the energy content of a class of errors in $(n-1)$ orthogonal directions. In the unlikely event that there is a value of ζ which minimizes these energy coefficients uniformly, this value of ζ would be that indicated by Eq. (17). A more typical situation would be one in which improvement in performance in one direction carries with it a concomitant degradation in performance in one or more of the other directions. The index H provides an ordering of possible independent variables in this more common circumstance satisfying some, but not all, of the desirata used by Müller and Weber[6] to delineate a measure of the quality of controllability.

To illustrate these notions, consider the following dynamic model of a re-entry vehicle whose primary motion is confined to the X-Z plane,

$$\dot{X} = V\cos\gamma \qquad \dot{Z} = V\sin\gamma \qquad \dot{\gamma} = AV^{-1} \qquad (18)$$

where X and Z are downrange position and altitude, respectively, in a target-centered coordinate system; γ is the flight-path angle; the nominal velocity $V(t)$ is given; and the acceleration $A(t)$ is the control instrument and is assumed to be slowly varying. Observe that Eq. (18) is a very simple representation of the actual equation of motion for the vehicle. Neglected are the nonlinearities, samplers, autopilot dynamics, and exogeneous influences which affect actual vehicle motion. The simplification implicit in Eq. (18) is intentional since one of the objectives of this study is to determine the degree to which simple analysis models can be used to derive controllers for a complicated dynamic system.

The class of transformations considered in this study are fixed rotations in the X-Z plane; i.e.,

$$T_R(\zeta) = \begin{bmatrix} T_{11} & 0 \\ 0 & 1 \end{bmatrix} \qquad T_{11} = \begin{bmatrix} \cos\zeta & \sin\zeta \\ -\sin\zeta & \cos\zeta \end{bmatrix} \qquad (19)$$

Although this class of transformations is obviously restricted, it has considerable appeal. First and foremost is interest that stems from the requirement that the guidance law be easy to implement. This means that the independent variable must be easily computed in terms of the measured trajectory variables, and that the dynamic equation of the vehicle in the perceived coordinate system must be amenable to analysis. The first of these desirata is obviously satisfied by $T_R(\zeta)$. If $\zeta = 0$, the independent variable is downrange position, and if $\zeta = 90$ the independent variable is altitude. Both of these are assumed to be directly measured quantities. Equation (19) thus gives additional flexibility with little additional complexity in computing x_{p1}.

There are other independent variables which are excluded from consideration that have obvious intuitive attraction. For example, pathlength-to-go and total energy appear to be potentially attractive variables not included within the class of transformations given by Eq. (19). Careful though suggests that such variables violate the restrictions which limit the choice of x_{p1} in Eq. (8). Thus, a pathlength-to-go depends not only on the present state, but also on the disturbances encountered on the remaining portion of the trajectory. This is not to say that an approximation to pathlength-to-go could not be produced which might yield an effective control law. Indeed, Speyer and Bryson[2] did something similar. Still, such an approach is distinct from that given in Eq. (8). Here the measured value of a trajectory variable like altitude is used as the variable of evolution and no approximations are needed for its evaluation. Even for general rotations in Eq. (19) the equations describing the evolution of the vehicle state with

§To call E as defined by Eq. (12) the control energy is something of a misnomer insofar as the $(t_2 - t_1)^{-1}$ factor gives E units more akin to power. Still, in this application it is useful to think of E as an energy figure with an implicit normalization with respect to the time increment.

¶As indicated in the Appendix, $E\{\Pi\alpha_i(\zeta)\}$ is used when $x(t_1)$ is random.

respect to the desired x_{p1} appear to be far simpler than would be the similar equations in which one tried to use such a variable as pseudopathlength-to-go as the independent variable.

With T_R (ζ) defined in Eq. (19) it follows directly that

$$\dot{x}_p(\zeta)=\begin{bmatrix} V\cos(\gamma-\zeta) \\ V\sin(\gamma-\zeta) \\ AV^{-1} \end{bmatrix} \quad f_r=\begin{bmatrix} 1 \\ \tan(\gamma-\zeta_0) \\ AV^{-2}\sec(\gamma-\zeta_0) \end{bmatrix} \tag{20}$$

where it has been assumed that x_p is such that $x_{p1}(\zeta)$ is monotone.

To find $\{\alpha_i\}$, the system controllability matrix must first be found. This matrix is given in Eq. (A2).

$$C_2=(M_0,\ M_1) \tag{21}$$

where [see Eq. (A4)]

$$M_0=\begin{bmatrix} 0 \\ 0 \\ V^{-2}\sec(\gamma-\zeta) \end{bmatrix} \quad M_1=\begin{bmatrix} 0 \\ V^{-2}\sec^3(\gamma-\zeta_0) \\ \cdots \end{bmatrix} \tag{22}$$

Using Eq. (22) it follows that

$$\eta_1=\|\eta_1\|\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \eta_2=\|\eta_2\|\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{23}$$

Equation (23) requires careful interpretation. The controllable subspace of the perturbation equation associated with Eq. (20) can be decomposed into an "easy" direction, η_1, and a "hard" direction, η_2. From Eq. (18) it is evident that the easy direction is associated with an angular error and the hard direction with a position error. This result is intuitively appealing since the acceleration acts directly on γ and only indirectly on position errors.

To compute the docility index given by Eq. (16) using the result given in Eq. (A14) only the appropriate amplitude normalization is yet to be determined. Equation (A10) provides an equation relating the time-based error x and the error perceived by the controller $x_r(\zeta)$. Substituting the required quantities into this equation

$$x_r(\zeta)=\begin{bmatrix} 0 & 0 & 0 \\ -\sin\gamma\sec(\gamma-\zeta) & \cos\gamma\sec(\gamma-\zeta) & 0 \\ -AV^{-2}\cos\zeta\sec(\gamma-\zeta) & -AV^{-2}\sin\zeta\sec(\gamma-\zeta) & 1 \end{bmatrix} x \tag{24}$$

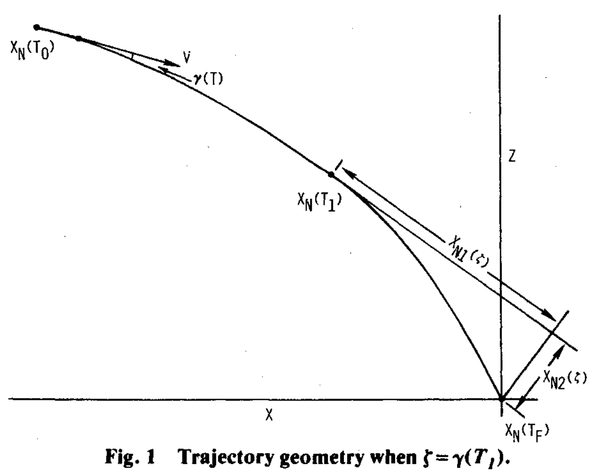

Fig. 1 Trajectory geometry when $\zeta=\gamma(T_1)$.

In the application of interest, the primary initial errors are in position since the angular orientation of the vehicle is controlled quite accurately during boost and free flight. Suppose, therefore, that the initial error in the $(x_n,\ u_n)$ coordinate system is given by

$$x=\begin{bmatrix} \delta X \\ \delta Z \\ \delta\gamma \end{bmatrix}=\begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix} \tag{25}$$

The parameter λ gives the direction of the initial error and the size of this error has been normalized. The perceived error resulting is

$$x_r(\zeta)=\begin{bmatrix} 0 \\ \sec(\gamma-\zeta)\sin(\lambda-\gamma) \\ AV^{-2}\sec(\gamma-\zeta)\cos(\lambda-\zeta) \end{bmatrix} \tag{26}$$

Assume that the initial error angle λ is uniformly distributed on $[0, 2\pi]$ and is independent of γ. From Eq. (A13)

$$\begin{aligned} \|\eta_1\| &= |AV^{-1}\sec(\gamma-\zeta)\cos(\lambda-\gamma)| \\ \|\eta_2\| &= |\sec(\gamma-\zeta)\sin(\lambda-\zeta)| \end{aligned} \tag{27}$$

Substituting these values into the defining equation for H

$$H(\zeta)=(A^2V^{-2}/24)\,[1+2\sin^2(\gamma-\zeta)]\sec^2(\gamma-\zeta) \tag{28}$$

From Eq. (28) it follows that the best choice of ζ at time t_1 would be $\zeta=\gamma(t_1)$;

$$H(\zeta=\gamma(t_1))=\min_{\zeta} H(\zeta)=(A^2V^{-2}/24) \tag{29}$$

From Eq. (29) it is clear that the locally best coordinate system in which to control the vehicle is one in which the instantaneous position is expressed in the rotated coordinate system shown in Fig. 1. The independent variable $x_{p1}(\zeta^*)$ can be thought of as a range variable and the component of state in which error can be measured can be thought of as a pseudomiss variable. Although Eq. (29) would appear to suggest that good performance could be attained by setting $\zeta(t)=\gamma(t)$, a time-variable rotation is not permitted by the hypotheses which lead to Eq. (29). The rotation angle must be constant throughout the trajectory and as a consequence a judicious choice for ζ would be that which approximates most closely the realized value of γ for that portion of the trajectory of primary concern. On a mission in which terminal miss is a prime performance contributor, $\zeta=\gamma_n(t_f)$ would appear to be a rational choice.

An Example

To explore some of the nuances of the synthesis procedure presented in the foregoing sections, a simulation study was performed to test some of the guidance laws previously described on a sophisticated and relatively complete simulation model of a particular aerodynamically controlled re-entry vehicle. There were a number of questions to which this study gave at least partial answer. Of most concern were the relative merits of a time-based guidance law and one which used a trajectory variable as the variable of evolution. Second, the correspondence of the docility index given by Eq. (16) and the observed behavioral qualities of the guidance law were of interest. Finally, the general question of the utility of the analysis model, Eq. (18), in constructing guidance laws was also under investigation.

There are four guidance laws whose behavior was studied in some detail. They are: 1) U_t – time is the independent variable; 2) U_X – downrange position is the independent variable ($\zeta=0$); 3) U_z – altitude is the independent variable ($\zeta=\pi/2$); and 4) U_* – $\zeta=\gamma_n(t_f)$.

The regulators U_X and U_Z are of interest because of their implementational simplicity. In both cases a directly measured trajectory variable is used as the independent variable. On the other hand, the independent variable associated with U_* must be computed from measured quantities. Still, U_* is of significant interest because it locally minimizes H at the termination of the trajectory. The regulator U_t is explicitly time dependent and its performance forms a base of comparison for the other regulators.

The weighting matrices in the performance indices for the regulators were selected to penalize each of the regulators similarly for similar errors. First, consider (P, Q, R) in Eq. (3). The weighting on control was constant and was essentially given by

$$R = 1/\Delta A^2_{\max} \tag{30}$$

where $\Delta A_{\max}$ is the maximum permissible magnitude of variation of the acceleration from its nominal value. The state-error weighting took the form

$$Q(t) = \text{diag}[q_1^2(t), \quad q_2^2(t), \quad q_3^2(t)] \tag{31}$$

The state weights were

$$q_i^2(t) = 1/(\Delta x_i^2)_{\max} \tag{32}$$

where the allowable position error decreased monotonically from the order of 10^4 ft at re-entry to the order of 10 ft at impact. The allowable angular deviation was also time variable but was monotonically increasing. The terminal position weight, P, is given by

$$P = Q(t_f) \tag{33}$$

The weighting matrices for U_X, U_Z, and U_* were defined similarly. In each case the associated Q was diagonal. Because of the degeneracy of the state space, q_1 is irrevelant and the complete position error is weighted by q_2. For this reason

$$Q_\zeta = \left(\frac{\mathrm{d}x_{p1}(\zeta)}{\mathrm{d}t}\right)^{-1}(0, q_1^2 + q_2^2, q_3^2) \tag{34}$$

where q_i is given by Eq. (32) and the first factor in Q_ζ is a time normalization; the other performance weights are

$$R_\zeta = \left(\frac{\mathrm{d}x_{p1}(\zeta)}{\mathrm{d}t}\right)^{-1} R$$

$$P_\zeta = Q_\zeta[x_{p1}(t_f; \quad \zeta)] \tag{35}$$

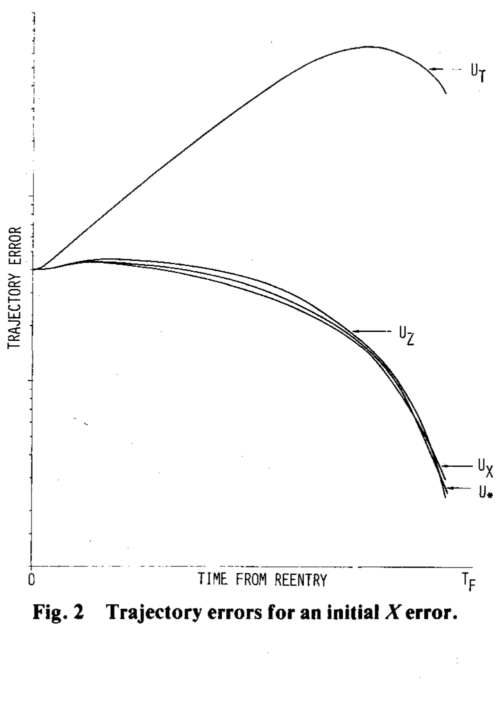

Fig. 2 Trajectory errors for an initial X error.

Equation (34) makes the position and angular error weights in the $x_r(\zeta)$ coordinate system compatible with those used in deriving U_t. Note that U_X, U_Z, and U_* measure position error as a scalar while U_t senses a two-dimensional position error. The weighting matrices given by Eqs (30-35) provide like weights to like errors and the influence on performance due to time-scale distortion is avoided by the "velocity" factor in Eqs. (34) and (35).

To relate the docility index given in Eq. (16) to actual vehicle performance, a simulation study was undertaken. The vehicle simulation equations provided a detailed description of the dynamic structure of an actual re-entry vehicle. This comprehensive model actually provided an impediment to good performance for the regulators designed here because these regulators were based upon a dynamic hypothesis that was deficient in many respects. Of the simulation results obtained, those from three numerical experiments are presented here. In each case the nominal trajectory began with

$$X(t_0) = \text{order } 10^5 \text{ ft}$$

$$Z(t_0) = \text{order } 10^5 \text{ ft}$$

$$t_0 = 0; \ t_f = \text{order } 10 \text{ s}$$

$$\gamma(t_0) \in [0, \pi/4); \gamma(t_f) \in (\pi/4, \pi/2)$$

The three tests are described as follows: 1) $X_p(t_0) - X_n(t_0) = -$order of 10^3 ft, $x_i(t_0) = 0$ otherwise; 2) $Z_p(t_0) - Z_n(t_0) = -$order of 10^3 ft, $x_i(t_0) = 0$ otherwise; and 3) no initial error, 0.9 times nominal air density on $[0, t_f/2]$, 1.1 times nominal air density on $(t_f/2, t_f)$.

Figure 2 shows the result of the first test. For an initial X perturbation the perpendicular path errors are plotted on a log scale. The error magnitudes have been normalized, and while the relative errors of the regulators are accurate, their absolute values have no significance. All three time-independent controllers behave in the way one would expect. All begin with the same trajectory error and in each case the error builds up slightly because of autopilot effects. Since

$$|\gamma_n(t_0)| < |\gamma_n(t_0) - \gamma_n(t_f)| < |\gamma_n(t_0) - \pi/2| \tag{36}$$

on the trajectory of interest, one would expect from Eq. (28) that the preference ordering of the regulators would initially be U_X first, U_* second, and U_Z third. This is indeed the case, as shown in Fig. 2. For that part of the trajectory satisfying Eq. (36), the docility index provides an ordering in accord with the trajectory-following fidelity of the associated regulator. Although the initial portion of the trajectory is subject to aberrant drag forces and acceleration limits, the slowly varying control gains tend to reduce sampler and autopilot effects. The dynamic equation given by Eq. (18) is a fully adequate regulator synthesis model for 90% of the trajectory. The comparison of the time-independent controllers at termination is obscured by autopilot influences. Near the end of the trajectory, the regulator gains are rapidly varying, and the autopilot has difficulty in providing a faithful reproduction of the required actuating signal. Even here (see Table 1) the comparison of U_Z and U_* with U_X is that predicted on the basis of Eq. (28). The former regulators have impact errors that are within the best accuracy to be expected, while U_X has a somewhat larger error.

The performance of U_t as given in Fig. 2 and Table 1 appears superficially to be incorrect. Far from causing a diminution of the initial error, U_t causes the trajectory-following error to increase by an order of magnitude. The error at impact is inferior to that obtainable with no feedback regulator at all. Another way of comparing U_t with the set of U_r is in terms of the amount of control used on the trajectory. In a guidance system using a performance index like Eq. (3), the regulator seeks to use as little control force as possible while simultaneously maintaining good trajectory-following qualities. The magnitude function $|R^{-1}GKX|$ is a measure of

Table 1 Guidance-law performance

Closed-loop regulator	Perturbations	ΔDR, ft	Performance deviations Error in flight time, % of nominal	$\int \lvert u_i \rvert d\tau^{a} / \int \lvert u_t \rvert d\tau$
U_t	Initial downrange	1572	0.031	1
	Initial altitude	3199	0.34	1
	Density	325	−0.87	1
U_X	Initial downrange	6.3	−0.11	1.45×10^{-2}
	Initial altitude	16.2	−0.69	5.35×10^{-3}
	Density	9.8	−0.76	7.7×10^{-2}
U_z	Initial downrange	0.3	−0.17	9.3×10^{-3}
	Initial altitude	0.4	−0.70	6.9×10^{-3}
	Density	5.0	−0.77	5.7×10^{-2}
U_*	Initial downrange	2.0	−0.15	1.07×10^{-2}
	Initial altitude	2.9	−0.64	5.1×10^{-3}
	Density	4.8	−0.77	5.9×10^{-2}

[a] See Eq. (41).

the degree of apprehension with which the regulator views its instantaneous state. Thus,

$$\psi(\zeta) = \int_{t_0}^{t_f} |R_r' G_r K_r X_r| \, dt \tag{37}$$

is a measure of disapprobation for the closed-loop vehicle trajectory. Defining $\psi(t)$ in the obvious way, ψ provides an indication of how control intensive each of the regulators is. The actual value of the actuating signal was not used as the integrand in Eq. (37) because saturation in the actuators tends to desensitize this index. From Table 1, where the relative values of ψ are given, the regulator U_t is seen to use a factor of 10^2 more control than is needed by the time-independent controllers. The excessive use of acceleration on the part of U_t coupled with trajectory-following performance is approximately 10^2 worse than that attained with the U_r regulators.

The reason for the conspicuous inferiority of U_t lies in the way the trajectory-following problem is posed. The time-based regulator tries not to minimize the true trajectory error, but rather moves to correct the error measured by x in Eq. (2). For the test shown in Fig. 2 the initial X error was such as to initiate re-entry at a point closer to the target than the nominal starting point. Because the controller has no way of slowing the vehicle directly, U_t reacts to that portion of the error that is inherently a time translation by increasing the pathlength of the perturbed trajectory. Increasing pathlength is control-energy intensive and tends to cause large errors normal to the trajectory. This "time-equivalent" bubble is characteristic of time-based regulators and is not present in the response of the modified LQ regulators. Table 1 indicates that U_t is able to achieve much tighter control over time of flight than can any of the U_r. Unfortunately, this attribute is not of any particular advantage in this mission.

Figure 3 shows the trajectory bubble for an initial Z error. Only the nominal and the U_t trajectory are shown. The time-independent controllers would be indistinguishable on the scale of this drawing. Figure 3 is not shown to exact scale but is indicative of qualitative features shown by the actual vehicle trajectories. As before, a small initial error is caused to grow by U_t in order to slow the effective forward velocity. As was the case with an initial X error, U_t overcompensates for the initial time translation and has a perturbed flight time greater than the nominal value. This timing error is presumably due to controller saturation near impact.

The improvement factors associated with the time-independent controllers are repeated in this example. The impact error of U_t is orders of magnitude greater than that accruing to the alternative regulators. The improvement in control utilization is again on the order of 10^2. Although not shown in Fig. 3, the relative performance of U_X, U_Z, and U_* was in accordance with that predicted on the basis of the docility index H. The previously encountered difficulty with U_X near impact manifests itself again. The excellent performance of U_Z should be considered to be more a function of fortuitous circumstance than design.

The final example provides an interesting assessment of the robustness of the guidance laws studied here. In this simulation there was no initial error, but the dynamic equation of the vehicle was changed by decreasing air density by 10% on the first half of the re-entry trajectory and increasing it by 10% on the last half. Ideally, the regulator output should be nearly zero since there are only slight path-following errors created by the open-loop portion of the guidance law. The three time-independent guidance laws do follow the path quite closely, albeit at a different rate than does the nominal. The related errors are uniformly less than 10 ft. On the other hand, as shown in Fig. 4, U_t finds the density variation particularly bewildering. In the low-density portion of the flight a large error builds up as U_t tires to slow the vehicle by increasing path length. When the sign of the density changes, U_t must now increase its speed along the nominal path starting with what is now a sizable state error. It does this in part by crossing over the nominal path and impacting the ground short of the target. Because of this terminal maneuver, the factor by which U_t deteriorates performance is less than that found in some of the earlier tests.

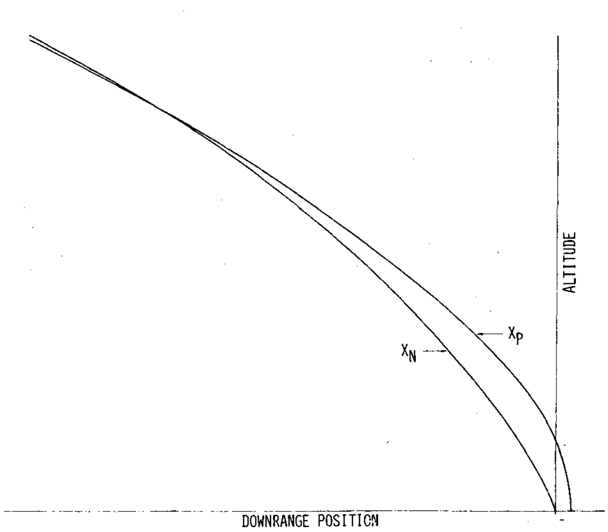

Fig. 3 Vehicle trajectory for initial Z error and time as the independent variable.

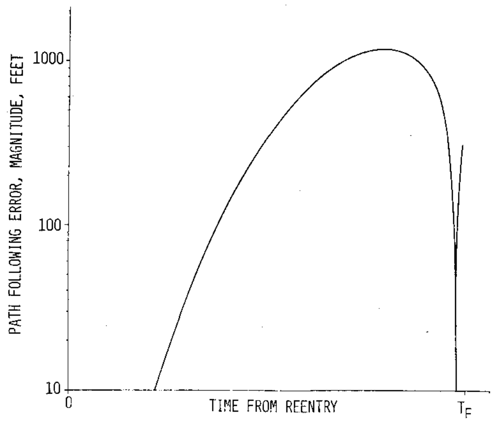

Fig. 4 Trajectory errors for density variation and time as the independent variable.

One peculiarity of the path flown by U_t is the fact that the time of flight differs from nominal to a greater degree than do the other regulators. This may be due to the disadvantageous state into which U_t forces the vehicle midway in the flight. As Table 1 shows, U_t uses excessive amounts of control force. As before U_Z and U_* are clearly superior to U_X.

While not exhaustive, this simulation study illustrates some of the behavioral anomalies of U_t. The time-independent regulators give performance that is several orders of magnitude superior to that attained by U_t. A comparison of U_X, U_Z, and U_* on portions of the trajectory where the neglected vehicle dynamics had least influence suggests that the docility index given by Eq. (28) is suitable in this application. Further study is necessary to resolve certain apparent aberrations in relative performance.

Conclusion

An aerodynamically controlled re-entry vehicle has dynamic pecularities that tend to discourage the use of LQ feedback regulators in guidance. The disadvantageous features of the vehicle stem largely from its weak controllability. By the simple artifice of using a trajectory variable in place of time as the independent variable of evolution, important deficiencies of the LQ regulator are avoided and a robust guidance law produced.

The selection of this independent variable from the available alternatives is complicated by the often contradictory exigencies of guidance-law simplicity and the dynamic response of the vehicle. Using a simplified analysis model, this paper provides an index of quality for the closed-loop response characteristics of the vehicle. This index is phrased in terms of the local energy content associated with perturbations from the nominal trajectory. In terms of this index it is possible to rank different choices for an independent variable.

Although the index was selected with a view toward maintaining a reasonable level of analytical tractability, it is still true that the form of H precludes the development of a direct algorithm for finding the best x_{p1}. Some intuitively appealing choices for an independent variable are fairly easily compared, and it has been shown that the "natural" independent variable Z has desirable closed-loop properties. For the trajectory considered here either U_Z or U_* would be adequate. The superiority of these regulators to the classical LQ regulator is readily apparent both in the fidelity of trajectory following and in the judicious use of available control resources. The performance of the time-based regulator is so poor that it does not appear to be a rational candidate for this type of reentry mission.

Appendix

The intent of the Appendix is twofold. First some properties of controllable linear systems are reviewd and it is shown that there are different degrees of local controllability. For a system with two-dimensional state space the relevant directions are given in Eq. (A6) and the corresponding measures of local controllability are given in Eqs. (A7) and (A8).

The second question addressed in the Appendix is that of evaluating the value of the docility index explicitly and using this index to compare performance when different independent variables are used in the model given by Eq. (18). The individual factors in H must first be derived and then normalized to account for distortions occasioned by the coordinate system changes.

The perturbation variables (x,u) of Eq. (3) are related by the differential equation [see Eq. (4)].

$$\dot{x}=Fx+Gu \qquad x(t_0)=x_0 \tag{A1}$$

Define the controllability matrix C_n by

$$C_n=[M_0, M_1, \ldots, M_{n-1}]$$
$$M_0=G$$
$$M_{k+1}=-FM_k+\dot{M}_k \qquad (k=1, \ldots) \tag{A2}$$

The system (F, G) is completely controllable if the rank of C_n is n. In this circumstance W [see Eq. (13)] is positive and the minimum-energy transfer from any initial state to the origin in the interval $[t_1, t_2]$ is that given by Eq. (12).[5]

It was shown in Ref. 1 that W can be expressed as the sum

$$W(t_0, t_1)=\sum_{i,j=0}^{\infty}\frac{(t_1-t_0)^{i+j+1}}{i!j!(i+j+1)}M_i(t_0)M_j'(t_0) \tag{A3}$$

If x_0 is a unit eigenvector of $W(t_0, t_1)$ with associated eigenvalue λ, then the energy required to drive x_0 to the origin is λ^{-1}.

Suppose (t_1-t_0) is small. In the system of interest u is scalar and M_i are column vectors. Retaining only third-order terms

$$W(t_0, t_1)=(t_1-t_0)M_0M_0'+\frac{(t_1-t_0)^2}{2}(M_0M_1'+M_1M_0')$$
$$+\frac{(t_1-t_0)^3}{3}\left[\frac{M_0M_2'}{2}+\frac{M_2M_0'}{2}+M_1M_1'\right]+o(t_1-t_0)^3 \tag{A4}$$

The eigenvectors $\{\eta_i(t_0, t_1)\}$ of the positive matrix W span R^n and can be found from the algorithm

$$\eta_1'W\eta_1=\max_{\eta}\eta'W\eta \qquad (\eta'\eta=1)$$
$$\eta_2'W\eta_2=\max_{\eta}\eta'W\eta \qquad \left\{\begin{array}{l}\eta'\eta=1\\ \eta'\eta_1=0\end{array}\right\}$$
$$\vdots$$

Let the associated eigenvalues be labeled $\lambda_1, \lambda_2, \ldots$. From Eq. (A4) it is clear that for small (t_1-t_0)

$$\eta_1\cong M_0\|M_0\|^{-1} \qquad \lambda_1=(t_1-t_0)\|M_0\|^2$$
$$E(\eta_1, \mathrm{d}t)=\|M_0\|^{-2}\mathrm{d}t^{-2} \tag{A5}$$

The energy associated with the next eigenvector η_2 can be similarly computed

$$\eta_2\cong C(M_1-M_1'M_0(M_0'M_0)^{-1}M_0)$$
$$E(\eta_2, \mathrm{d}t)=\tfrac{1}{3}(\eta_2'M_1M_1'\eta_2)^{-1}\mathrm{d}t^{-4} \tag{A6}$$

where C is a normalization constant.

This procedure can be continued, but Eqs. (A5) and (A6) will suffice for this problem. The directions η_1 and η_2 can be thought of as "easy" and "hard" directions to control, respectively. Since $M_0 = G$, the fact that $\eta_1 = CM_0$ simply expresses the fact that the first-order influence of u is in the direction G. It is more difficult to cause the system to move in the η_2 direction as evidenced by the fact that $E(\eta_2, \mathrm{d}t)$ is proportional to $\mathrm{d}t^{-4}$. The energy figure for this latter direction depends upon FG as well as the time variation of G.

The problem of using these energy figures to compare different independent variables is made difficult by structural degeneracies in the (F_r, G_r) system [see Eq. (8)]. Indeed, since $x_{r1} \equiv 0$, (F_r, G_r) can not be controllable in the usual sense. Consider the system described by Eq. (8). Mimicking the development leading to Eq. (A2), (F_r, G_r) will be said to be controllable if rank $C_{n-1} = n-1$,

$$C_{n-1} = \begin{bmatrix} 0 \ . \ . \ . \ 0 \\ \tilde{C}_{n-1} \end{bmatrix}$$

The controllability subspace would be all x_r orthogonal to $(1, 0, ..., 0)$.

The energy content in the directions η_1 and η_2 in the (x_r, u_r) coordinate system follow directly from Eqs. (A5) and (A6)

$$E(\eta_1;\, \mathrm{d}x_{p1}(\zeta)) = \|M_0\|^{-2} \|\eta_1\|^2 |\mathrm{d}x_{p1}(\zeta)|^{-2} \qquad \text{(A7)}$$

$$E(\eta_2;\, \mathrm{d}x_{p1}(\zeta)) = \frac{(\eta_2' M_1 M_1' \eta_2)^{-1}}{3\|\eta_2\|^2} \|\eta_2\|^2 |\mathrm{d}x_{p1}(\zeta)|^{-4} \qquad \text{(A8)}$$

From Eqs. (A7) and (A8) the energy content is the product of three types of factors. The first factor is the energy associated with a unit perturbation in the η_i direction. The second factor $\|\eta\|^2$ scales the error with respect to the actual perturbation expected in the indicated direction. The final factor scales the energy with respect to the increment in the independent variable.

As written, Eqs. (A7) and (A8) are not in a form which facilitates comparison of different values of ζ. Each of the factors in these equations depends on ζ and simple scale changes in x_{p1} are translated into apparent changes in energy content. To form a valid basis for comparison, Eqs. (A7) and (A8) must be expressed as a function of the same error in the time domain. From Eq. (1) it is clear that

$$\mathrm{d}x_{p1}(\zeta) = (T_R(\zeta) f(x_p, u_p))_1 \mathrm{d}t \qquad \text{(A9)}$$

Equation (A9) provides the factor required to normalize the time interval over which control is accomplished.

The normalization of the eigenvectors η_1 is somewhat more subtle. A given initial error in the x_n coordinate system is transformed into a different apparent error in the $x_r(\zeta)$-plane. Suppose there is an initial error x in the x_n-plane. This error becomes $x(\zeta)$ under the transformation $T_R(\zeta)$;

$$x(\zeta) = T_R(\zeta) X$$

It was shown in Ref. 4 that the perceived error when $x_{p1}(\zeta)$ is used as the variable of evolution is

$$x_r(\zeta) = T_r(x_{p1}(\zeta))\, T_R(\zeta) x \qquad \text{(A10)}$$

where

$$T_r = \begin{pmatrix} 0 & 0 & 0 \\ -f_{r2} & & \\ & I & \\ -f_{r3} & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \end{pmatrix} \qquad \text{(A11)}$$

An initial error x gives rise to an indicated error $x_r(\zeta)$ of the following form

$$x_r(\zeta) = \sum_{i=1}^{n-1} \frac{n_i' T_r T_R x}{\|n_i\|^2} \eta_i \qquad \text{(A12)}$$

The coefficients of the decomposition of the error vector given above measure the relative size of sensed errors in the different coordinate systems. In the problem under study here x is not known a priori and will be considered to be a random variable. In this case the relative sizes of the initial errors in the x_r coordinate system will be

$$\|\eta_i\| = \|\beta_i' T_r(x_{p1}(\zeta))\, T_R(\zeta) x\| \qquad (i = 1, \ . \ . \ . \ n-1) \qquad \text{(A13)}$$

where β_i is a unit vector in the η_i direction.

Combining Eqs. (A7), (A8), (A9), and (A13)

$$E(\eta_1;\mathrm{d}x_{p1}) = \|M_0\|^{-2} [\, (T_R f)_{1f6p1203-2} \|\beta_1' T_r T_R x\|^2 \mathrm{d}t^{-2}$$

$$E(\eta_2;\mathrm{d}x_{p1}) = \tfrac{1}{3} (\beta_2' M_1 M_1' \beta_2)^{-1} [\, (T_R f)_1]^{-4} \|\beta_2 T_r T_R x\|^2 \mathrm{d}t^{-4}$$

$$\vdots$$

For convenience let $E(\eta_i;\, \mathrm{d}x_{p1}) = \alpha_i \mathrm{d}t^{-2i+2}$ where α_i is defined above. With the energy content of errors in the η_i directions given by Eq. (A14), a scalar-valued measure of system docility would be a useful intermediary for comparing the relative merits of different values of ζ. Obviously, if there existed a value of ζ say ζ^* which minimized α_i uniformly;

$$\alpha_i(\zeta^*) \geq \min_{\zeta} \alpha_i(\zeta) \qquad (i = 1, ..., n-1)$$

then ζ^* would yield the best possible choice for $x_{p1}(\zeta)$. Unfortunately, a uniformly best value of ζ will seldom exist, and the analyst must be content with something less.

The docility index found most suitable for this class of problem is given by H;

$$H(\xi) = E\left\{ \sum_{i=1}^{n-1} \alpha_i(\zeta) \right\} \qquad \text{(A15)}$$

where $E\{\ \}$ denotes mathematical expectation. That value of ζ which minimizes $H(\zeta)$ is said to be the locally best choice of ζ;

$$H(\zeta^*) = \inf_{\zeta} H(\zeta) \qquad \text{(A16)}$$

Since α_i cannot be made uniformly small by selection of ζ, the expectation of the product of the α's is made small. Note that if for $\zeta = \zeta_0$ the controllability subspace of (F_r, G_r) is of dimension less than $n-1$, then $H(\zeta_0) = \infty$. It is interesting to note that $H(\zeta)$ bares close kinship with measures of controllability proposed in Refs. 6 and 7 which use det W^{-1} to induce a controllability ordering on the set of admissible controllers. Of course, the normalizations involved in deriving Eq. (A15) makes the final criterion somewhat different.

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